**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Calculus of variations

## Lecture 6. Lagrange problem with high derivatives

We have the method of the analysis for the minimization problem of the integral functional that depends from unknown function and its first derivative. This problem can be transformed to the second order Euler differential equation. If the functional depends from many unknown functions, we obtain the system of the analogical Euler equations as necessary conditions of minimum. However, there exist many practical problems, where the functional depends from high order derivatives of the unknown functions. We shell try to extend the previous results to these problems. The high order Euler–Poisson differential equation is the extension of the Euler equation to the Lagrange problem with high derivatives. We consider the problem of bending of the elastic beam as an example.

**6.1. Problem statement**

We would like to minimize the functional



where *F* is a given function, *r* is an order of the high derivative.

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| **Question**: *How many boundary conditions  is necessary for the analysis of this problem?* |

We would like to obtain the easiest Lagrange problem as a partial case of the new problem with  Besides, the boundary conditions must be symmetric with respect to the ends of the given interval. Therefore, the unknown function  satisfies the equalities

 (6.1)

 (6.2)

where  are given numbers,  Then, we have the following minimization problem.

**Problem 6.1**. *Find the function*  *that minimize the functional I with boundary conditions* (6.1), (6.2).

**6.2. Euler – Poisson Equation**

We try to extend the known ideas to this problem.

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| **Question**: *How we can solve this problem?* |

Let *u* be a solution of our problem. Then we have the inequality  for all function *v* that satisfies the boundary conditions (6.1), (6.2). Determine the function of one variable



where *σ* is a number, and *h* is a smooth enough function on the interval .

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| **Question**: *What are the boundary conditions for the function h?* |

We would like to guaranty that the function  is admissible. Therefore, the function *h* satisfies the homogeneous boundary conditions

 (6.3)

Then the function  is admissible, because satisfies the boundary conditions (6.1), (6.2).

The functional *I* has the minimum at the point *u* if and only if the number 0 is the point of the minimum for the function *f*. Therefore, we try calculating the derivative of the function *f* at zero and equaling it to 0.

Let the function *F* be smooth enough; and  are its corresponding partial derivatives.

**Lemma 6.1**. *The derivative of the function f at zero is equal to*

 (6.4)

**Proof**. We have



Using the Taylor formula, we get



where  as  Then we obtain



After dividing by *σ* and passing to the limit as  we get



Use the formula of the integration by parts. We have



because of the boundary conditions (6.3). Therefore, the previous equality can be transformed to (6.4).

The right side of the equality (6.4) is called the *variation of the functional I* at the point *u*with respect to the direction *h* and denoted by  Hence, the solution *u* of Problem 6.1 satisfies the equality

 (6.5)

for all function *h* that satisfies the boundary conditions (6.3).

**Theorem 6.1**. *The solution of Problem* 6.1 *satisfies Euler–Poisson equation*

 (6.6)

**Proof**. Using Lemma 6.1, we transform the equality (6.5) to



It is true for function *h*, which satisfies the boundary conditions (6.3). Using the Basic Lemma of Variational Calculus we obtain the equality (6.6).

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| **Question**: *What is the order of the equation* (6.6)? |

The solution of the Problem 6.1 can be found from the 2*r* differential equations (6.6) with boundary conditions

 (6.7)

 (6.8)

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| **Conclusion**: *Problem* 6.1 *can be solved with using  of the* 2*r order differential equations* (6.6) *with boundary conditions* (6.7), (6.8). |

Consider some examples.

**6.3. Example**

Minimize the functional



with boundary conditions



Determine the Euler–Poisson equation



We obtain the general its solution



The unknown constants can be found from the given boundary conditions. We have



Then





So  and  Hence, the minimum of the given functional is determined by the curve 

Find the corresponding value of the functional. We have  Then we find the value



**6.4. Bending of the elastic beam**

Consider the physical application of the obtained result. We have the elastic beam as a one-dimensional body with the space coordinate *x*. We would like to determine the form of the beam under a given force . Note that the force does not depend from the time. This phenomenon is described by the deviation . We try to determine this function by the principle of least action.

We consider the stationary process. The energy *E* of the beam is a sum of the interior energy *Ei* of the elasticity force and the exterior energy *Ee* of the given force *f*.

If the form of the beam is described by the function , then the interior energy is proportional to the square of the curvature *C* of the function *u*



where the constant *k* is characterized by the material of the beam. Let us consider the part of the curve (see Figure 6.1). The mean curvature of this part is the ratio between the increment Δ*α* of the angle of tangent line and the length Δ*s* of curve interval. Therefore, the curvature at the point *x* is the derivative

.



Figure 6.1. The curvature of the curve.

This formula can be transformed to the equality



It is known that the tangent of the tangent line angle at a point is the derivative of the function at this point, namely  Therefore,  Then we get



It is known the equality



Hence, the interior energy is determine by the formula



This is the value of the energy at the concrete point *x*,because the function *u* depends from *x.*

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| **Question**: *How we can determine the interior energy of the beam with length L*? |

The interior energy of the beam with length *L* is equal to the integral



The exterior energy is the work of this force with sign “minus” because the work is realized opposite with respect to the force. The work is the product between the force *f* and the deviation *u.* Therefore, the exterior energy of the beam with length *L* is equal to the integral



Then we have the total energy

 (6.9)

By the principle of least action, we minimize the energy *E*. The functional *E* depends from the second order derivative of the unknown function. Therefore, it is necessary four boundary conditions. Let the beam be fixed at the boundary. Then we have the boundary conditions

 (6.10)

We suppose also that the interior energy at the ends of the beam is absent. Therefore, we obtain two additional boundary conditions

 (6.11)

Hence, we have the problem of minimization of the functional (6.9) with boundary conditions (6.10), (6.11).

We solve this problem for the partial case. Let the exterior force be small enough. Therefore, the curvature will be square of the derivative  is negligible. Then the functional *E* can be transformed to the easier form

 (6.12)

In this case the function *F* of Problem 5.1 is determine by the formula



Then we have Euler –Poisson equation



So we have the fourth order differential equation



We can solve it with four boundary conditions (5.10), (5.11) for the concrete force function *f*.

### Outcome

* The solution of Lagrange problem with high derivatives of the unknown functions satisfies Euler – Poisson equation with boundary conditions.
* Euler – Poisson equation has the order 2*r* if *r* is the order of the high derivative of the problem statement.
* The solution of the obtained boundary problem can be the solution of Lagrange problem, but may be it is not its solution.
* The problem of bending of the elastic beam as an example of this theory.

### Task 5. Minimization of the functional that depends from the second derivative of the unknown function

Find the function  that minimize the integral  


with boundary conditions



The values of parameters.

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| --- | --- | --- | --- | --- | --- | --- | --- |
| Variant |  |  |  |  |  |  |  |
| 1 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 2 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 3 |  | -1 | 0 | -1 | 0 | -1 | 0 |
| 4 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 5 |  | 0 | 2 | 0 | -1 | 0 | 1 |
| 6 |  | -π | 0 | 0 | 1 | 1 | 0 |
| 7 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 8 |  | -1 | 1 | 1 | 0 | 1 | 0 |
| 9 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 10 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 11 |  | 0 | 1 | -1 | 0 | -1 | 0 |
| 12 |  | -1 | 0 | -1 | 0 | -1 | 0 |
| 13 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 14 |  | 0 | 1 | 0 | -1 | 0 | 1 |
| 15 |  | 0 | π | 0 | 1 | 1 | 0 |
| 16 |  | 0 | 1 | 0 | 1 | 0 | 1 |
| 17 |  | -1 | 1 | 1 | 0 | 1 | 0 |
| 18 |  | 0 | 1 | 0 | 1 | 0 | 1 |

Steps of the task.

1. Give the problem statement.
2. Determine the system of the Euler – Poisson equation.
3. Find the general solution of this equation.
4. Find the solution of the Euler – Poisson equation that satisfies given boundary conditions.
5. Show the graph of this solution.

**Literature**

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### Next step

We have the standard method for solving the problems of minimization for integral functionals with one or many unknown functions. These functionals can depend from the unknown functions and its derivatives. However, the unknown functions depend of one variable only. We will try to extend our results to the functions of many variables.